

# Dimensional reduction in causal set gravity

S. CARLIP\*

*Department of Physics  
University of California  
Davis, CA 95616  
USA*

## Abstract

Results from a number of different approaches to quantum gravity suggest that the effective dimension of spacetime may drop to  $d = 2$  at small scales. I show that two different dimensional estimators in causal set theory display the same behavior, and argue that a third, the spectral dimension, may exhibit a related phenomenon of “asymptotic silence.”

---

\*email: [carlip@physics.ucdavis.edu](mailto:carlip@physics.ucdavis.edu)

## 1. Introduction

In classical physics, the dimension of spacetime is a fixed parameter, specified from the outset. In quantum gravity, this may no longer be the case: dimension may be a quantum observable, taking different values under different circumstances. In particular, there are intriguing hints from a number of different approaches to quantum gravity that the effective dimension of spacetime drops to two at very small distances [1, 2]. This “spontaneous dimensional reduction” was first noted in high temperature string theory [3], and then a few years later in the discrete path integral approach of causal dynamical triangulations [4]. Since then, the same behavior has been seen in the asymptotic safety program [5, 6], the short distance approximation to the Wheeler-DeWitt equation [1], aspects of loop quantum gravity [7], some formulations of noncommutative geometry [8–10] or minimum length [11], and perhaps Hořava-Lifshitz gravity [12]. The generality of these results suggests that dimensional reduction may be a fundamental feature of quantum gravity.

One important approach to quantum gravity, however, seems to be an exception. In [13], Eichhorn and Mizera show that the spectral dimension of a causal set *increases* at short distances. In this paper, I will show that two other dimensional estimators for causal sets—the Myrheim-Meyer dimension of a small causal set and the dimension determined by the causal set Laplacian—display the more standard drop to  $d = 2$  and short distances. I will argue that the Eichhorn-Mizera result may have a different interpretation, as an indication of short distance “asymptotic silence” [14], a behavior that has also been associated with dimensional reduction [1, 2, 15].

This paper should be read as a report on work in progress. As we shall see, a number of relevant concepts (e.g., Hadamard Greens functions on causal sets) and calculations (e.g., Myrheim-Meyer dimension for “small” causal sets with more than six elements) do not yet exist. But the preliminary results are promising, and this seems to be a program worthy of further study.

## 2. Causal sets

A causal set [16] is a discrete spacetime in which events have prescribed causal relations. Such a set is characterized by a partial order  $\prec$  (where  $x \prec y$  means “ $x$  is to the past of  $y$ ”) satisfying

1. transitivity:  $x \prec y$  and  $y \prec z \Rightarrow x \prec z$ ;
2. acyclicity:  $x \prec y$  and  $y \prec x \Rightarrow x = y$ ;
3. local finiteness: for any  $x$  and  $y$ , the number of elements  $z$  such that  $x \prec z \prec y$  is finite.

Mathematically, these conditions define a locally finite partially ordered set, or poset. Physically, the causal relations should be thought of as determining “most” of the metric. Indeed, in the continuum, the causal structure of a globally hyperbolic manifold determines the metric up to a conformal factor [17]; in causal set theory, the missing conformal factor is simply the number of points in a region.

Causal sets with clear physical meaning can be constructed by randomly “sprinkling” points on a fixed spacetime. Given a globally hyperbolic manifold  $M$  with metric  $g$ , select a set of points by a Poisson process so that the probability of finding  $m$  points in any region of volume  $V$  is

$$P_V(m) = \frac{(\rho V)^m}{m!} e^{-\rho V} \quad (2.1)$$

for some discreteness scale  $\rho^{-1}$ . Assign to these points the causal relations determined by the metric  $g$ , and then “remove” the manifold  $M$ , leaving only the set of points and relations. At

scales larger than  $\rho^{-1}$ , the resulting causal set is believed to approximate  $M$  well. In particular, if  $M$  is Minkowski space, the causal set preserves statistical Lorentz invariance [18], a highly nontrivial characteristic for any discretization.

### 3. Myrheim-Meyer dimension of a small causal set

As in other discrete approaches to quantum gravity, it is not obvious what one means by the “dimension” of a causal set. For a space with an analog of a Riemannian metric, a popular choice is the spectral dimension, but it is not obvious that this is appropriate to a Lorentzian spacetime; I will return to this issue later. For a causal set, the most common choice for a dimensional estimator is the Myrheim-Meyer dimension [19, 20], which is based on a count of the number of causally related points.

More precisely, let us start with a causal set derived from a Poisson sprinkling of points in  $d$ -dimensional Minkowski space. Choose an Alexandrov interval, or “causal diamond,”  $\mathcal{A}$ , that is, the intersection of the future of some point  $p$  and the past of another point  $q$ . Let  $\langle C_1 \rangle$  be the average number of points in  $\mathcal{A}$ , and  $\langle C_2 \rangle$  be the average number of causal relations, that is, pairs  $x, y$  such that  $x \prec y$ .  $\langle C_1 \rangle$  and  $\langle C_2 \rangle$  depend on the volume of  $\mathcal{A}$  and the discreteness scale  $\rho^{-1}$ , but a suitable ratio depends only on the dimension:

$$\frac{\langle C_2 \rangle}{\langle C_1 \rangle^2} = \frac{\Gamma(d+1)\Gamma(\frac{d}{2})}{4\Gamma(\frac{3d}{2})} \quad (3.1)$$

For an arbitrary causal set, the Myrheim-Meyer dimension  $d_M$  is then defined as the value  $d$  for which (3.1) holds. One can also consider a sprinkling of points in a curved spacetime; if the curvature is small, a generalization of (3.1) involving chains of three and four related points can eliminate distortions due to curvature [21].

We are interested here in the dimension of “small” causal sets. There are several different things this might mean:

1. One might simply take a random causal set with a small number  $C_1$  of elements. As a practical matter,  $C_1$  must be *very* small: the number of distinct causal sets with  $C_1$  elements goes as  $2^{C_1^2/4}$ , and the causal sets have only been fully enumerated up to  $C_1 = 16$  [22].
2. For larger  $C_1$ , random causal sets are dominated by Kleitman-Rothschild, or KR, orders [23, 24]. These are three-layered posets with approximately  $C_1/4$  elements in the first and third layers and  $C_1/2$  elements in the second; an element in the first or third layer is causally related to about half of the elements in the second layer, and almost every element in the first layer is related to almost every element in the third. Numerical studies indicate that these sets become important at  $C_1 \sim 50$  [24]. While KR orders must be dynamically suppressed at large scales if causal set theory is to reproduce anything like our universe, it is plausible that they remain important at reasonably small scales.
3. The preceding criteria do not include dynamics, in part because the dynamical behavior of causal set theory is not well understood. One might, however, consider random sprinklings of points in known spacetimes—Minkowski space, for instance—and look at their small scale behavior.

The first two of these approaches show clear signs of dimensional reduction. For example, suppose we start with a large causal set and chose a subset containing four elements. There

are a total of 16 possible causal structures among those elements, having between zero and six causal relations. If these structures occur with equal probability, the average  $\langle C_2 \rangle$  is  $\frac{13}{4}$ , and the Myrheim-Meyer dimension (3.1) is 2.27. For random causal sets with four, five, or six elements, as enumerated in the Chapel Hill poset atlas [25], the Myrheim-Meyer dimensions range from 2.15 to 2.27. Similarly, for a random KR order, the dimension is 2.38.

For the third approach, more numerical work is needed. But Reid has looked at random sprinklings in Minkowski space [26], and the results show a decrease in the Myrheim-Meyer dimension to a bit less than 2 for small subintervals, as expected in short distance dimensional reduction.

## 4. Laplacians and Greens functions

Consider a massless field in a  $d$ -dimensional spacetime. At short distances, the Hadamard Greens function takes the form

$$G^{(1)}(x, x') \sim \begin{cases} \sigma(x, x')^{-(d-2)/2} & d > 2 \\ \ln \sigma(x, x') & d = 2 \end{cases} \quad (4.1)$$

where Synge's world function  $\sigma(x, x')$  is half the squared geodesic distance between  $x$  and  $x'$ . The dimension is thus determined, in a manifestly physical way, by the rate at which the two-point function blows up at coincident points.

As usual, it is not immediately obvious how to extend this expression to a discrete spacetime. Recently, however, considerable progress has been made in defining Laplacians and retarded Greens functions on causal sets obtained by random sprinklings of points in Minkowski space in two [27], four [28], and arbitrary [29, 30] dimensions. While the retarded Greens functions are not the same as the Hadamard functions (4.1), they still provide use useful information.

Aslanbeigi et al. have examined the behavior of these quantities averaged over causal sets obtained by sprinklings on  $d$ -dimensional Minkowski space [30]. The averaged Laplacians have plane wave eigenfunctions  $e^{ip \cdot x}$ , as expected from Poincaré invariance, with calculable eigenvalues  $g(p)$ . Hence

$$G_R(x, x') = \int_{\mathcal{C}} d^d p g(p)^{-1} e^{ip \cdot (x - x')} \quad (4.2)$$

In the IR limit relevant for long distance behavior,  $g(p)^{-1} \sim 1/p^2$ , confirming that the causal set Laplacians approximate the standard continuum operators. In the UV, though, one finds that

$$g(p)^{-1} \sim \alpha + \beta(p \cdot p)^{-d/2} \quad (4.3)$$

For the contour  $\mathcal{C}$  appropriate for a retarded Greens function, the integral (4.2) near the coincidence limit  $\sigma \rightarrow 0$  gives a delta function plus a finite correction, the normal behavior for a retarded Greens function [30]. But if, as in the continuum case, the Hadamard function can be obtained by choosing a different contour in (4.2), then (4.3) will lead to a Hadamard function  $G^{(1)} \sim \ln \sigma$  at short distances, the standard form for a two-dimensional massless field theory,\* although one may worry whether this reduction occurs below the discreteness scale.

To be confident of this claim, one would have to construct the full analog of the Hadamard Greens function in causal set theory and examine its UV limit. Recent work on field theory on causal sets [31, 32] suggests an approach to this problem, and work is in progress. But as in the preceding section, we already see strong hints of dimensional reduction.

---

\*A similar phenomenon occurs in Hořava-Lifshitz gravity [12], but in contrast to that model, the causal set result does not require a violation of Lorentz invariance.

## 5. Spectral dimension and asymptotic silence

Consider a random walk on a  $d$ -dimensional manifold with a Riemannian metric. Diffusion from an initial position  $x$  to a final position  $x'$  in a time  $s$  is described by a heat kernel  $K(x, x'; s)$ , which behaves for small  $s$  as [4]

$$K(x, x'; s) \sim (4\pi s)^{-d/2} e^{-\sigma(x, x')/2s} (1 + \mathcal{O}(s)) \quad (5.1)$$

In particular, the return probability  $K(x, x; s)$  is determined by the dimension. By generalizing (5.1) to an arbitrary space, discrete or continuous, on which a random walk can be defined, one obtains an effective dimension, the spectral dimension.

For several approaches to quantum gravity, including causal dynamical triangulations [4] and asymptotic safety [5], the spectral dimension exhibits short distance dimensional reduction to  $d = 2$ . For causal set theory, though, it does not. On the contrary, the spectral dimension increases at short distances [13]. What should one make of this?

Eichhorn and Mizera argue in [13] that the peculiar behavior of causal set theory comes from the Lorentzian signature of the metric, which in many cases leads to a “radical nonlocality”—a typical point can have infinitely many nearest neighbors, points connected by a single causal link. Now, as stressed in [1, 2], the importance of spectral dimension comes in part from the fact that Greens functions can be obtained as Laplace transforms of the heat kernel: (4.1) is a Laplace transform of (5.1). But for causal sets, the Greens functions of [27–30] contain nonlocal corrections, and the direct connection to the heat kernel for a random walk may be broken.

The results of [13] could, however, have a different implication. In a Lorentzian setting, a high spectral dimension—especially a high value of the “causal spectral dimension” of [13]—implies a suppression of the probability that two random walkers will meet within a given diffusion time. The observed rapid rise in spectral dimension at very short distances thus suggests that “nearby” points are increasingly causally disconnected. A very similar behavior occurs in cosmology near a spacelike singularity, where it is known as “asymptotic silence” [14]. As I first pointed out in [1], this phenomenon, which leads to locally Kasner-like behavior of the metric, might explain dimensional reduction: at certain scales,  $d$ -dimensional Kasner space has an effective dimension of two [33].

It should be possible to test this conjecture more directly. In the continuum, asymptotic silence is an “anti-Newtonian” limit, in which the speed of light goes to zero and nearby spacelike separated points become (nearly) causally disconnected. In the causal set context, defining “nearby” is nontrivial, but not impossible [34], and one can measure the minimum number of links  $N$  required for two nearby points to share a common point in the future. The short distance asymptotic silence conjecture is that while  $N$  should behave classically for pairs of points with large spatial separations, it should become much larger than its classical value as the spatial distance shrinks. If this is the case, the arguments of [1] would again predict spontaneous dimensional reduction.

## 6. Conclusion

While the evidence for short distance dimensional reduction in quantum gravity is far from conclusive, there are enough hints from enough different approaches to make the phenomenon at least plausible. But details remain elusive. We do not even know whether dimensional reduction is mainly kinematical or whether it depends sensitively on the dynamics: in the asymptotic safety scenario, for instance, the mere existence of a non-Gaussian UV fixed point is enough to indicate

two-dimensional behavior [6], while in some approaches based on noncommutative geometry the nature of dimensional reduction depends sensitively on a choice of deformed Laplacian [10].

Causal set theory offers a promising avenue to explore these issues. Much of what we know about causal sets is nondynamical, and there are several approaches to the dynamics that may not be equivalent [35]. While this paper is a start, there is clearly much more to be done:

- A systematic study of the Myrheim-Meyer dimension of small subsets of random sprinklings on various known manifolds could reveal more about the influence of large scale spacetime geometry, and thus dynamics, on small scale dimension. One might also look at the curvature-corrected dimensional estimator introduced in [21].
- A construction of the causal set Hadamard function, perhaps following [31, 32], and a study of its asymptotics in the manner of [30], would tell more reliably whether Greens functions exhibit dimensional reduction. One might also compute the heat kernels, and through that the spectral dimensions, of the Laplacians in [30].<sup>†</sup>
- More direct tests of short distance asymptotic silence would certainly be illuminating.

It may be that different dimensional estimators give different answers, and the full picture might require a better understanding of the quantum dynamics of causal sets. But the preliminary indications of short distance dimensional reduction in causal set theory seem promising.

## Acknowledgments

This work was supported in part by Department of Energy grant DE-FG02-91ER40674.

## References

- [1] S. Carlip, in *Proc. of the 25th Max Born Symposium: The Planck Scale*, AIP Conf. Proc. 1196 (2009) 72, arXiv:0909.3329.
- [2] S. Carlip, in *Proc., Foundations of Space and Time*, edited by J. Marugan, A. Weltman, and G.F.R. Ellis (Cambridge: Cambridge University Press, 2009), arXiv:1009.1136.
- [3] J. J. Atick and E. Witten, Nucl. Phys. B310 (1988) 291.
- [4] J. Ambjørn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. 95 (2005) 171301, arXiv:hep-th/0505113.
- [5] M. Reuter and F. Saueressig, JHEP 1112 (2011) 012, arXiv:1110.5224.
- [6] R. Percacci and D. Perini, Class. Quant. Grav. 21 (2004) 5035, arXiv:hep-th/0401071.
- [7] L. Modesto, Class. Quant. Grav. 26 (2009) 242002, arXiv:0812.2214.
- [8] D. Benedetti, Phys. Rev. Lett. 102 (2009) 111303, arXiv:0811.1396.
- [9] K. Nozari, V. Hosseinzadeh, and M. A. Gorji, arXiv:1504.07117.
- [10] M. Arzano and T. Trzesniewski, Phys. Rev. D89 (2014) 124024, arXiv:1404.4762.

---

<sup>†</sup>Just after this preprint first appeared, a preprint by Belechia et al. [36] answered this last question. The heat kernels of the nonlocal Laplacians [30] do, in fact, lead to a spectral dimension that falls to  $d = 2$  at short distances.

- [11] L. Modesto and P. Nicolini, Phys. Rev. D81 (2010) 104040, arXiv:0912.0220.
- [12] P. Hořava, Phys. Rev. Lett. 102 (2009) 161301, arXiv:0902.3657.
- [13] A. Eichhorn and S. Mizera, arXiv:1311.2530.
- [14] J. M. Heinzle, C. Uggla, and N. Röhr, Adv. Theor. Math. Phys. 13 (2009) 293, arXiv:gr-qc/0702141.
- [15] S. Carlip, R. A. Mosna, and J. P. M. Pitelli, Phys. Rev. Lett. 107 (2011) 021303, arXiv:1103.5993.
- [16] L. Bombelli, J. Lee, D. Meyer, and R. Sorkin, Phys. Rev. Lett. 59 (1987) 521.
- [17] D. Malament, J. Math. Phys. 18 (1977) 1399.
- [18] L. Bombelli, J. Henson, and R. D. Sorkin, Mod. Phys. Lett. A24 (2009) 2579, arXiv:gr-qc/0605006 .
- [19] J. Myrheim, 1978 CERN preprint TH-2538.
- [20] D. A. Meyer, Ph.D. thesis, MIT (1989), <http://hdl.handle.net/1721.1/14328>.
- [21] M. Roy, D. Sinha, and S. Surya, Phys. Rev D87 (2013) 044046 arXiv:1212.0631.
- [22] G. Brinkmann and B. D. McKay, Order 19 (2002) 147.
- [23] D. Kleitman and B. L. Rothschild, Trans. Am. Math. Soc. 205 (1975) 205.
- [24] J. Henson, D. P. Rideout, R. D. Sorkin, and S. Surya, arXiv:1504.05902.
- [25] C. A. Gann and R. A. Proctor, Chapel Hill Poset Atlas, <http://www.unc.edu/~rap/Posets/>.
- [26] D. D. Reid, Phys. Rev. D67 (2003) 024034, arXiv:gr-qc/0207103.
- [27] R. D. Sorkin, in *Approaches to quantum gravity*, edited by D. Oriti (Cambridge University Press, 2009) 26, arXiv:gr-qc/0703099.
- [28] D. M. T. Benincasa and F. Dowker, Phys. Rev. Lett. 104 (2010) 181301, arXiv:1001.2725.
- [29] F. Dowker and L. Glaser, Class. Quant. Grav. 30 (2013) 195016, arXiv:1305.2588.
- [30] S. Aslanbeigi, M. Saravani, and R. D. Sorkin, JHEP 1406 (2014) 024, arXiv:1403.1622.
- [31] S. Johnston, Phys. Rev. Lett. 103 (2009) 180401, arXiv:0909.0944.
- [32] A. Belenchia, D. Benincasa, and S. Liberati, JHEP 1503 (2015) 036, arXiv:1411.6513.
- [33] B. L. Hu and D. J. O'Connor, Phys. Rev. D34 (1986) 2535.
- [34] D. Rideout and P. Wallden, Class. Quant. Grav. 26 (2009) 155013, arXiv:0810.1768.
- [35] P. Wallden, J. Phys. Conf. Ser. 453 (2013) 012023.
- [36] A. Belenchia, D. Benincasa, A. Marcianò, and L. Modesto, arXiv:1507.00330.